Sum, Difference, and Double-Angle Identities

The sum and difference identities are used to simplify expressions and to determine the exact trigonometric values of some angles.

SUM IDENTITIES	Examples:
$\sin(A+B) = \sin A \cos B + \cos A \sin B$ $\cos(A+B) = \cos A \cos B - \sin A \sin B$ $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$	$\sin(12° + 23°) = \sin 12° \cos 23° + \cos 12° \sin 23°$ $\cos\left(\dfrac{\pi}{6} + \dfrac{\pi}{4}\right) = \cos\dfrac{\pi}{6}\cos\dfrac{\pi}{4} - \sin\dfrac{\pi}{6}\sin\dfrac{\pi}{4}$ $\tan(40° + 25°) = \dfrac{\tan 40° + \tan 25°}{1 - \tan 40° \tan 25°}$
DIFFERENCE IDENTITIES	Examples:
$\sin(A-B) = \sin A \cos B - \cos A \sin B$ $\cos(A-B) = \cos A \cos B + \sin A \sin B$ $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$	$\sin(52° - 33°) = \sin 52° \cos 33° - \cos 52° \sin 33°$ $\cos\left(\dfrac{\pi}{3} - \dfrac{\pi}{4}\right) = \cos\dfrac{\pi}{3}\cos\dfrac{\pi}{4} + \sin\dfrac{\pi}{3}\sin\dfrac{\pi}{4}$ $\tan(70° - 35°) = \dfrac{\tan 70° - \tan 35°}{1 + \tan 70° \tan 35°}$

Example 1: Simplify Expressions Using Sum and Difference Identities

Write each expression as a single trigonometric function and determine the exact value for each expression.

a. $\cos 290° \cos 80° + \sin 290° \sin 80°$

b. $\sin\left(\dfrac{5\pi}{12}\right)\cos\left(\dfrac{\pi}{3}\right) + \cos\left(\dfrac{5\pi}{12}\right)\sin\left(\dfrac{\pi}{3}\right)$

Solution:

$\cos 290° \cos 80° + \sin 290° \sin 80°$	$\sin\left(\dfrac{5\pi}{12}\right)\cos\left(\dfrac{\pi}{3}\right) + \cos\left(\dfrac{5\pi}{12}\right)\sin\left(\dfrac{\pi}{3}\right)$

Example 2: Determine Exact Trigonometric Values for Angles

Find the exact value for each of the following:

$\sin(195°)$	$\cos\left(\dfrac{5\pi}{12}\right)$

Example 3: Prove an Identity Using the Compound Angle Identities

Prove each of the following identities.

a. $\sin\left(\dfrac{\pi}{2} - x\right) = \cos x$

LHS	RHS
$\sin\left(\dfrac{\pi}{2} - x\right)$	$\cos x$

b. $\sin\left(x + \dfrac{\pi}{3}\right) - \cos\left(x + \dfrac{\pi}{6}\right) = \sin x$

LHS	RHS
$\sin\left(x + \dfrac{\pi}{3}\right) - \cos\left(x + \dfrac{\pi}{6}\right)$	$\sin x$

c. $\dfrac{\sin(a+b)}{\cos a \cos b} = \tan a + \tan b$

LHS	RHS
$\dfrac{\sin(a+b)}{\cos a \cos b}$	$\tan a + \tan b$

Example 4: Solve an Equation Involving the Compound Angle Identities

Solve $\sin x \cos\dfrac{\pi}{6} - \cos x \sin\dfrac{\pi}{6} = \dfrac{1}{2}, 0 \le x \le 2\pi$

Solution:

$\sin x \cos\dfrac{\pi}{6} - \cos x \sin\dfrac{\pi}{6} = \dfrac{1}{2}, 0 \le x \le 2\pi$

Example 5: Using Compound Angle Identities

If $\cos\theta = \dfrac{-3}{5}$ and $\dfrac{\pi}{2} < \theta < \pi$, determine the value of $\sin\left(\theta + \dfrac{\pi}{6}\right)$.

Solution:

DOUBLE ANGLE IDENTITIES	Examples:
$\sin 2A = 2\sin A\cos A$ $\cos 2A = \cos^2 A - \sin^2 A$ $\cos 2A = 2\cos^2 A - 1$ $\cos 2A = 1 - 2\sin^2 A$ $\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}$	$\sin\dfrac{\pi}{4} = 2\sin\dfrac{\pi}{8}\cos\dfrac{\pi}{8}$ $\cos 140° = \cos^2 70° - \sin^2 70°$ $\cos 140° = 2\cos^2 70° - 1$ $\cos 140° = 1 - 2\sin^2 70°$ $\tan\dfrac{\pi}{6} = \dfrac{2\tan\dfrac{\pi}{12}}{1 - \tan^2\dfrac{\pi}{12}}$

Example 6: Simplify Expressions Using Double-Angle Identities

Write each expression as a single trigonometric function and determine the exact value for each expression.

a. $\cos^2\dfrac{\pi}{3} - \sin^2\dfrac{\pi}{3}$

b. $2\sin\dfrac{\pi}{12}\cos\dfrac{\pi}{12}$

Solution:

$\cos^2\dfrac{\pi}{3} - \sin^2\dfrac{\pi}{3}$	$2\sin\dfrac{\pi}{12}\cos\dfrac{\pi}{12}$

Example 7: Simplify Expressions Using Identities & Recognizing Non-Permissible Values

Consider the expression $\dfrac{1-\cos 2x}{\sin 2x}$

    a. Determine the non-permissible values for the expression.
    b. Simplify the expression.

Solution:

a. $\sin 2x \neq 0$ This means that $2\sin x \cos x \neq 0$ So, $\quad \sin x \neq 0 \qquad \cos x \neq 0$ $\sin x = 0$ when $x = 0 \pm \pi n, \ n \in W$ $\cos x = 0$ when $x = \dfrac{\pi}{2} \pm \pi n, \ n \in W$ Combining these values means $x \neq \pm \dfrac{\pi}{2}n, \ n \in W$	b. $\dfrac{1-\cos 2x}{\sin 2x}$

Example 8: Prove an Identity Using a Double Angle Identity

Prove the following identities.

    a. $\dfrac{\sin 2x}{1+\cos 2x} = \tan x$

    b. $\dfrac{\cos 2x}{\cos x - \sin x} = \cos x + \sin x$

    c. $\tan 2x - 2\tan 2x \sin^2 x = \sin 2x$

    d. $\dfrac{1-\cos 2x}{\sin x \sin 2x} = \sec x$

Solution:

a. $\dfrac{\sin 2x}{1 + \cos 2x} = \tan x$

$\dfrac{\sin 2x}{1 + \cos 2x} = \tan x$	
LHS	RHS
$\dfrac{\sin 2x}{1 + \cos 2x}$	$\tan x$

b. $\dfrac{\cos 2x}{\cos x - \sin x} = \cos x + \sin x$

$\dfrac{\cos 2x}{\cos x - \sin x} = \cos x + \sin x$	
LHS	RHS
$\dfrac{\cos 2x}{\cos x - \sin x}$	$\cos x + \sin x$

c. $\tan 2x - 2\tan 2x \sin^2 x = \sin 2x$

$\tan 2x - 2\tan 2x \sin^2 x = \sin 2x$	
LHS	RHS
$\tan 2x - 2\tan 2x \sin^2 x$	$\sin 2x$

d. $\dfrac{1 - \cos 2x}{\sin x \sin 2x} = \sec x$

$\dfrac{1 - \cos 2x}{\sin x \sin 2x} = \sec x$	
LHS	RHS
$\dfrac{1 - \cos 2x}{\sin x \sin 2x}$	$\sec x$

Example 9: Solving Equations Using the Double Angle Identity

Solve for θ in general form and then in the specified domain. Use exact solutions when possible.

a. $\cos 2\theta + 1 - \cos\theta = 0, \ 0 \le \theta < 2\pi$
b. $\sin 2\theta + \sin\theta = 6\cos\theta + 3, \ -180° \le \theta \le 180°$
c. $3\tan\theta = \tan 2\theta, \ 0 \le \theta < 2\pi$

Solution:

a. $\cos 2\theta + 1 - \cos\theta = 0, \ 0 \le \theta < 2\pi$

b. $\sin 2\theta + \sin\theta = 6\cos\theta + 3; -180° \le \theta \le 180°$

c. $3\tan\theta = \tan 2\theta, \ 0 \le \theta < 2\pi$