Logarithmic and Exponential Equations

Solving Exponential Equations

One way to solve an exponential equation is to rewrite it so that both sides of the equation are powers with the same base. Then you can create another simpler equation by setting the exponents equal to each other and solving.

However, not all exponential equations can be solved this easily. Consider the equation $10^x = 13$. We cannot change both sides of the equation to the same base by simple inspection. So what do we do?

Solve $25^{x+1} = 125$
Express both sides in terms of the same base.
⇩
$5^{2(x+1)} = 5^3$
$5^{2x+2} = 5^3$
⇩
Equate the exponents and solve.
$2x + 2 = 3$
$x = \frac{1}{2}$

Solve $13 = 10^x$
Rewrite 13 as a power with base 10.
$\log 13 = 1.113943$
⇩
$10^{1.113943} = 10^x$
$1.113943 = x$

To rewrite 13 as a power with a base of 10, we need to use logarithms. Remember a log is just a name for an exponent that a specific base is raised to get an answer. Use your calculator to find log 13. Note the log key on your calculator is only for base 10.

But what about an equation like $5^{2x-1} = 3$?

Calculators don't have base 5 or base 3 log keys, so we have to change both bases to 10. This means we find the common log of 5 and the common log of 3.

$\log 5 = 0.698970$

⇩

This means $10^{0.698970} = 5$

$\log 3 = 0.477121$

⇩

This means $10^{0.477121} = 3$

A more formal method for solving this problem is presented next.

Solve $5^{2x-1} = 3$
Find the log of 5 and of 3, express with base 10.
⇩
$(10^{0.69897})^{2x-1} = 10^{0.477121}$
Since the bases are the same, equate the exponents.
⇩
$0.69897(2x-1) = 0.477121$
Solve for the variable.
⇩
$2x - 1 = 0.682606$
$2x = 1.682606$
$x = 0.841303$

Solve $5^{2x-1} = 3$
Take the common log of both sides.
$\log 5^{2x-1} = \log 3$
Remember the power rule for logs
$(2x-1)\log 5 = \log 3$
Solve for the variable.
$2x - 1 = \dfrac{\log 3}{\log 5}$
$2x = \left(\dfrac{\log 3}{\log 5} + 1\right)$
$x = \dfrac{1}{2}\left(\dfrac{\log 3}{\log 5} + 1\right)$
$x = 0.841303$
You can check your solution by substituting 0.841303 back into the original equation.

NOTE: Given that c, L, R > 0 and $c \neq 1$ (The logarithm of 0 or a negative number is undefined):

If L = R, then $\log_c L = \log_c R$

So, you can solve an exponential equation algebraically by

- taking logarithms of both sides of the equation, then
- applying the power law for logarithms to solve for an unknown.

Example 1: Solve Exponential Equations Using Logarithms

Solve. Express your answer as an exact value and as a decimal correct to two decimal places.

a. $5^x = 200$	b. $3(8^{2x-3}) = 45327$	c. $6^{3x+1} = 8^{x+3}$

Example 2: Model Exponential Growth

A town has a current population of 12 468. The population is growing by 2% per year.

a. Write an exponential equation to model the population growth.
b. What will be the town's population in eight years?
c. When will the population first reach 20 000?

Solution:

Example 3: Model Exponential Decay

The half-life of a substance is 12.5 hours. If the initial amount is 50 g, how long will it take to reduce to 23 g?

Solution:

Example 4: Extension

Two die-cast metal cars were purchased at the same time. The first cost $99.50 and appreciated by 10% every 3 years. The second cost $49.50 and appreciated 16% every 2 years. Determine how many years it will take for the two cars to be of equal value.

Solution:

Solving Logarithmic Equations

Some log equations can be solved by converting them into exponential form and then solving.

Solve $\log_5 x = 3$
Convert to an exponential equation and solve.
⇩
$5^3 = x$
$125 = x$
Check your answer.

Solve $\log_2(3x) = \log_2(2x+7)$
Since the logs are equal and have the same base, then the arguments are equal. Solve.
⇩
$3x = 2x + 7$
$x = 7$
Check your answer.

Others can be solved using the following property:

Given that c, L, R > 0 and $c \neq 1$:

If $\log_c L = \log_c R$, then L = R

To solve most logarithmic equations:

1. Isolate the logarithmic expression. You may need to use the laws of logarithms to create one logarithmic term.
2. Rewrite in the equation in exponential form.
3. Solve for the variable.
4. Identify whether any roots are extraneous by substituting into the original equation and determining whether or not all the logarithms are defined.

Solve $\log_2(15x-1) - \log_2(x-1) = 4$
Use the laws of logarithms to create one logarithmic term.
⇩
$\log_2\left(\frac{15x-1}{x-1}\right) = 4$
Convert to an exponential equation and solve.
⇩
$\frac{15x-1}{x-1} = 2^4$
$15x - 1 = 16(x-1)$
$15x - 1 = 16x - 16$
$-x = -15$
$x = 15$
Check your answer.

Example 5: Solve Logarithmic Equations

Solve. Remember to check for extraneous roots.

a. $\log_4 9 = x$ b. $\log_4(5x+1) = \log_4(x+17)$ c. $\log(5x) - \log(x-1) = 1$ d. $\log_6(x-3) + \log_6(x+6) = 2$

Solution:

a. $\log_4 9 = x$	b. $\log_4(5x+1) = \log_4(x+17)$	c. $\log(5x) - \log(x-1) = 1$
Shortcut: Change of Base Formula $\log_b M = \dfrac{\log M}{\log b}$ Check your solution in the original equation.	Check your solution in the original equation.	Check your solution in the original equation.

d. $\log_6(x-3) + \log_6(x+6) = 2$

Check your solutions in the original equation.

Example 6: Model a Situation Using a Logarithmic Equation

Paleontologists can estimate the size of a dinosaur from incomplete skeletal remains. For a carnivorous dinosaur, the relationship between the length, s, in metres, of the skull and the body mass, m, in kilograms, can be expressed using the logarithmic equation:

$3.6022 \log s = \log m - 3.444$

To the nearest hundredth of a centimetre, what was the skull length of a Tyrannosaurus Rex with an estimated body mass of 5500 kg?

Solution: