Analyzing Rational Functions

VERTICAL ASYMPTOTES AND POINTS OF DISCONTINUITY

Graphs of rational functions have a variety of shapes and features. We have seen that the graphs of rational functions, $f(x) = \frac{p(x)}{q(x)}$, are *discontinuous* for any non-permissible values of x (values of x that make $q(x) = 0$). One of the possible features that correspond to a non-permissible value of x is a vertical asymptote. Another possible feature that corresponds to a non-permissible value of x is a ***point of discontinuity.***

Point of Discontinuity

- A point, described by an ordered pair, at which the graph of the function is not continuous.
- Results in a single point missing from the graph, which is represented by an open circle.

Example 1: Graph a Rational Function with a Point of Discontinuity

Sketch the graph of $f(x) = \frac{x^2 - 3x - 4}{x - 4}$. Analyze its behaviour near its non-permissible value.

Solution:

Factor the numerator and denominator, then simplify. $f(x) = \frac{x^2 - 3x - 4}{x - 4} = \frac{\cancel{(x-4)}(x+1)}{\cancel{(x-4)}}$ Factor

Notice we have the same factor in both numerator and denominator

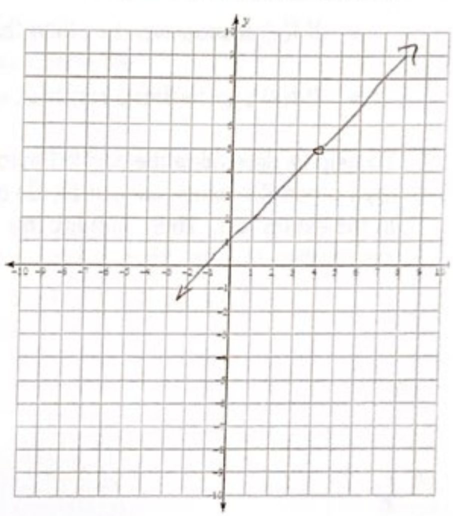
The non-permissible value of x is x = 4.

Note that the simplified function is a linear function. (The x-4 would cancel)

Complete the table of values to observe the function's behaviour near its non-permissible value of x=4:

x	3.5	3.9	3.99	4	4.01	4.1	4.5
f(x)	4.5	4.9	4.99	DNE	5.01	5.1	5.5

From the table we can see that the value of f(x) gets closer and closer to 5 as x gets closer to 4 from *either* side. This value is known as **the limit** of the function as x approaches 4. A limit is the y value of a function that is approached as x approaches a certain value.



- Recall that for any *rational* function, f(x), if f(x) approaches $\pm\infty$ as x approaches (from either side) a non-permissible value, then there will be a *vertical asymptote* at that non-permissible value.
- If f(x) approaches (from either side) a *limit* as x approaches a non-permissible value, then there will be a *point of discontinuity* at that non-permissible value of x.

So, for this example, there is a Point of Discontinuity at (4 , 5) *

* To easily determine the y-coordinate of the POD, substitute the x-coordinate of the POD into the *simplified* function and evaluate:

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Determining Vertical Asymptotes and Points of Discontinuity

To quickly determine if the graph of a rational function will have a vertical asymptote, a point of discontinuity, or both, you begin by factoring the numerator and denominator. There will be a:

- **Vertical Asymptote** if the factor that corresponds to the non-permissible value appears in the denominator to a *greater* degree than in the numerator.
- **Point of Discontinuity** if the factor that corresponds to the non-permissible value appears in the denominator to an *equal or lesser* degree than in the numerator.
- **Examples:**

POD if denominator is cancelled by numerator. VA if it is not.

$f(x)=\dfrac{(x-1)(x+3)}{(x+1)}$ has a VA at $x=-1$

$g(x)=\dfrac{(x+2)}{(x+2)(x+2)}$ has a ~~POD~~ VA at $x=-2$

$h(x)=\dfrac{(x-1)(x+4)}{(x+4)}$ has a POD at $x=-4$

$k(x)=\dfrac{(x+2)(x-3)}{(x+1)(x-3)}$ has a POD at $x=3$ and a VA at $x=-1$

HORIZONTAL ASYMPTOTES

To determine whether or not a rational function has a horizontal asymptote, consider the *end behaviour* of the function (that is, the behaviour of f(x) as x approaches $\pm\infty$).

- If f(x) approaches $\pm\infty$, then there is no horizontal asymptote.
- If f(x) approaches a constant, c, then there is a horizontal asymptote at y = c.

To help us determine the end behavior of the function, and, subsequently the equation of the horizontal asymptote, if it exists, we can divide both *numerator* and *denominator* by the highest power of x that appears in the expression. Then, simplify this expression and examine the behavior of the function as x approaches $\pm\infty$. In other words, evaluate the limit of the function as $x\to\pm\infty$.

Examples:

Determine the equation of the horizontal asymptote, if it exists, for each function:

$f(x)=\dfrac{8x^3+2x+3}{4x^3+5x^2}$ - Bullet 1
- Divide leading coefficients

See bullet points below to help with this question

$HA = \frac{8}{4} = 2$

$g(x)=\dfrac{2x^2+3x}{4x^3-1}$ Bullet 2. The degree of the denominator is higher than the degree of the numerator, so $\boxed{HA = 0}$

$h(x)=\dfrac{x^3+x+9}{2x^2+1}$ Bullet 3 - The degree of the numerator is greater than the degree of the denominator, so there is $\boxed{\text{no HA}}$

This procedure can be simplified to quickly determine the equation of the horizontal asymptote:

1. • If the numerator and denominator have the *same* degree and the leading coefficients are, respectively, *a* and *b*, then the horizontal asymptote is given by $y=\dfrac{a}{b}$.

   (For $f(x)=\dfrac{8x^3+2x+3}{4x^3+5x^2}$ the horizontal asymptote is at y = 2)

2. • If the degree of the ***denominator*** is greater than that of the numerator, then the horizontal asymptote is given by y = 0.

   (For $g(x)=\dfrac{2x^2+3x}{4x^3-1}$ the horizontal asymptote is at y = 0)

3. • If the degree of the ***numerator*** is greater than that of the denominator, then the function will approach $\pm\infty$ and there will be ***no*** horizontal asymptote.

   (For $h(x)=\dfrac{x^3+x+9}{2x^2+1}$ there is no horizontal asymptote)

Example 2: Sketch the Graph of a Rational Function

Sketch the graphs of the following functions:

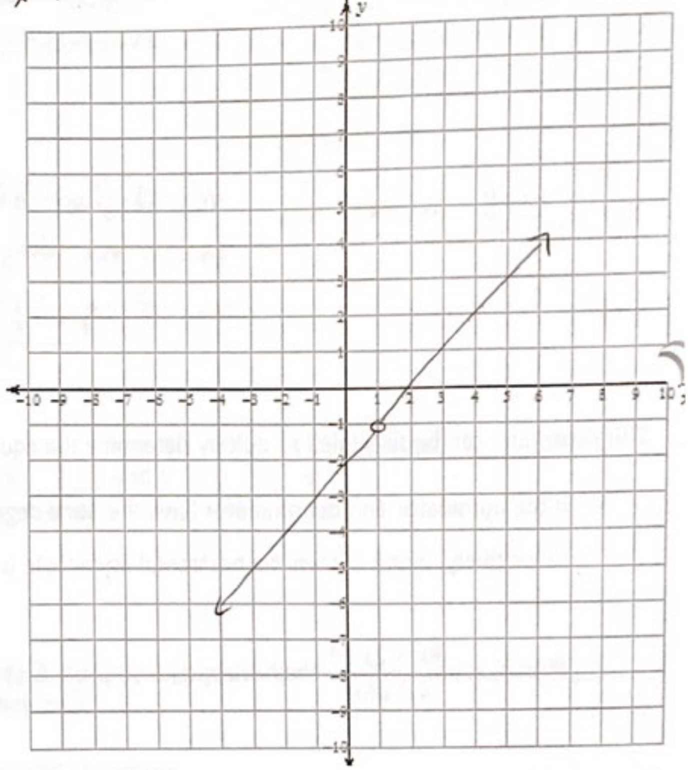
a. $f(x)=\dfrac{x^2-3x+2}{x-1}$ b. $f(x)=\dfrac{x^2+2x-8}{x^2+5x+4}$

Solution:

Express each function in factored form. Sketch the graphs using x-and y-intercepts, asymptotes, points of discontinuity, and any other necessary key points. Remember you can only cross the x- or y-axis at an intercept.

a. $f(x)=\dfrac{x^2-3x+2}{x-1} = \dfrac{(x-2)\cancel{(x-1)}}{\cancel{(x-1)}} = x-2$

x-intercepts	$x = 2$
y-intercept	-2
Vertical asymptotes	None
Points of Discontinuity	$x = 1$
Horizontal asymptote	None
Other key points	N/A
Domain	$x \neq 1, x \in R$
Range	$y \neq -1, y \in R$



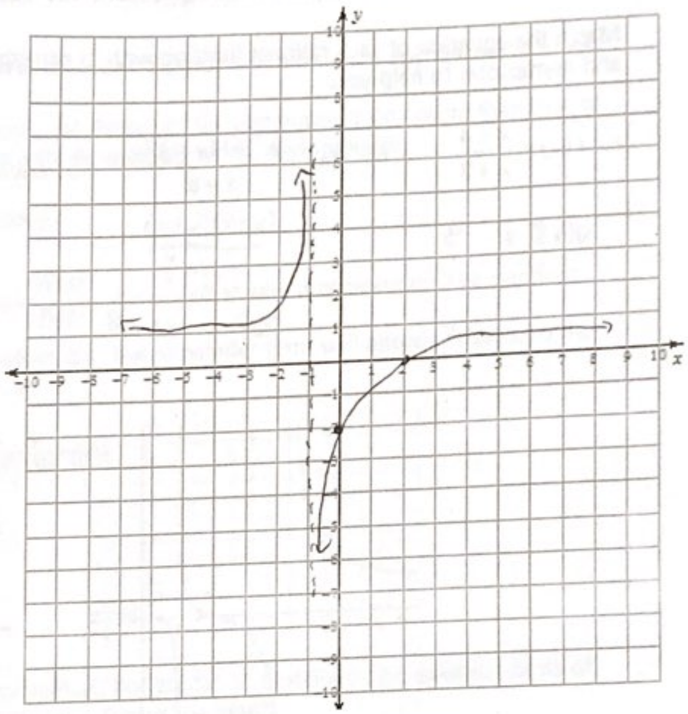
To find x-intercepts → it is when the numerator would equal 0

y-int → set $x = 0$

There is a point of discontinuity at $x = 1$, because that value is cancelled by the numerator

No horizontal asymptote because the degree of the numerator is greater than the degree of the denominator.

b. $f(x)=\dfrac{x^2+2x-8}{x^2+5x+4}$



x-intercepts	$x=2$
y-intercept	$y=-2$
Vertical asymptotes	$x=-1$
Points of Discontinuity	$x=-4$
Horizontal asymptote	$y=1$
Other key points	N/A
Domain	$x\neq-4,-1, x\in\mathbb{R}$
Range	$y\neq1, 2, x\in\mathbb{R}$

Behaviour near Vertical Asymptotes:

Factor: $\dfrac{x^2+2x-8}{x^2+5x+4}$

$\dfrac{\cancel{(x+4)}(x-2)}{\cancel{(x+4)}(x+1)}$

$=\dfrac{x-2}{x+1}$

VA at $x=-1$ because $(x+1)$ is not cancelled by the numerator.

POD at $x=-4$ because $(x+4)$ is cancelled by the numerator.

HA at $y=1$ because the degrees are the same and the leading coefficients are both 1.

Example 3: Match Graphs and Equations for Rational Functions

Match the equation of each rational function with its corresponding graph. Use key features such as intercepts and asymptotes to help you.

a. $f(x) = \dfrac{x+4}{x+8}$

VA @ $x = -8$

b. $f(x) = \dfrac{x^2+12x+32}{x+8}$

$\dfrac{(x+8)(x+4)}{(x+4)}$

PoD at $x = -8$

c. $f(x) = \dfrac{x^2+12x+32}{x^2+10x+16}$

$\dfrac{(x+8)(x+4)}{(x+8)(x+2)}$

VA at $x = -2$

PoD at $x = -8$

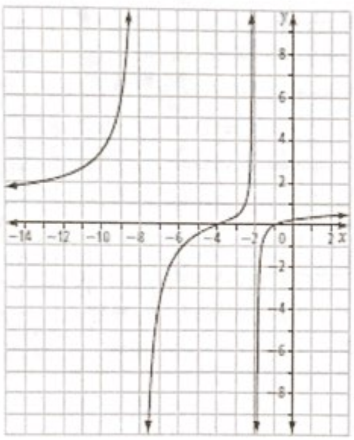
d. $f(x) = \dfrac{x^2+5x+4}{x^2+10x+16}$

$\dfrac{(x+4)(x+1)}{(x+8)(x+2)}$

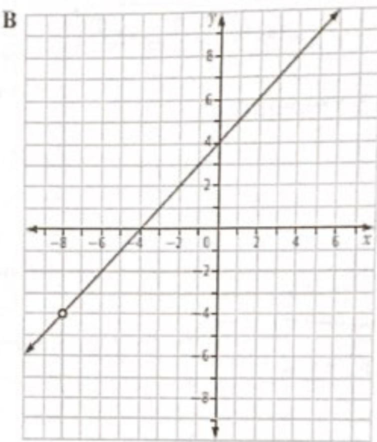
VA at $x = -8$

VA at $x = -2$

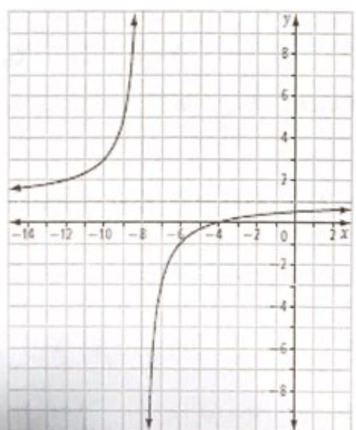
A



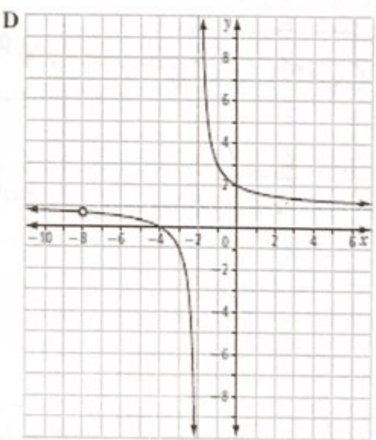
B



C



D



a. C

b. B

c. D

d. A